# Construction of Miniversal Deformations of Lie Algebras

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## Introduction

In this paper we consider deformations of finite or infinite dimensional Lie algebras over a field of characteristic 0. By "deformations of a Lie algebra" we mean the (affine algebraic) manifold of all Lie brackets. Consider the quotient of this variety by the action of the group GL. It is well-known (see [Hart]) that in the category of algebraic varieties the quotient by a group action does not always exist. Specifically, there is in general no universal deformation of a Lie algebra $L$ with a commutative algebra base $A$ with the property that for any other deformation of $L$ with base $B$ there exists a unique homomorphism $f: B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $L$ with base $A$ *versal*.

Classical deformation theory of associative and Lie algebras began with the works of Gerstenhaber [G] and Nijenhuis-Richardson [NR] in the 1960's. They studied one-parameter deformations and established the connection between Lie algebra cohomology and infinitesimal deformations. They did not study the versal property of deformations.

A more general deformation theory for Lie algebras follows from Schlessinger's work [Sch]. If we consider deformations with base spec $A$, where $A$ is a local algebra, this set-up is adequate to study the problem of "universality" among formal deformations. This was worked out for Lie algebras in [Fi1], [Fi3]; it turns out that in this case under some minor restrictions there exists a so-called *miniversal* element. The problem is to construct this element.

There is confusion in the literature when one tries to describe *all* the nonequivalent deformations of a given Lie algebra. There were several attempts to work out an appropriate theory for solving this basic problem in deformation theory, but none of them were completely adequate.

The construction below is parallel to the general constructions in deformation theory, as in [P], [I], [La], [GoM], [K]. The general theory, which can provide a construction of a local miniversal deformation, is outlined in [Fi1]. The procedure however needs a proper theory of Massey operations in the cohomology, and an algorithm for computing all the possible ways for a given infinitesimal deformation to extend to a formal deformation. The proper theory of Massey operations is developed in [FuL]. Our understanding of the construction arose from the study of the infinite dimensional Lie algebra $L_1$ of polynomial vector fields in $\mathbb{C}$ with trivial 1-jet at 0, in which case we completely described a miniversal deformation. In [FiFu] we proved that the base of the miniversal deformation of this Lie

algebra is the union of three algebraic curves: two smooth curves and another curve with a cusp at 0, with the tangent lines to all three curves coinciding at 0.

The structure of the paper is as follows: In Section 1 we give the necessary definitions and some facts on infinitesimal deformations. In Section 2 we recall Harrison cohomology, in Section 3 discuss obstruction theory. Section 4 gives the theoretical construction of a miniversal deformation, and some preliminary computations. Section 5 recalls the proper Massey product definition and describes its properties (see [FuL]). In Section 6 we calculate obstructions. Section 7 provides a scheme for computing the base of a miniversal deformation of a Lie algebra convenient for practical use. In Section 8 we apply the construction to the Lie algebra $L_1$.

## 1. Lie algebra deformations

**1.1.** Let $L$ be a Lie algebra over a characteristic 0 field $\mathbb{K}$, and let $A$ be a commutative algebra with identity over $\mathbb{K}$ with a fixed augmentation $\varepsilon\colon A \to \mathbb{K}$, $\varepsilon(1) = 1$; we set $\mathrm{Ker}\,\varepsilon = \mathfrak{m}$. To avoid transfinite induction, we will assume that $\dim(\mathfrak{m}^k/\mathfrak{m}^{k+1}) < \infty$ for all $k$.

DEFINITION 1.1. A *deformation* $\lambda$ of $L$ with base $(A, \mathfrak{m})$, or simply with base $A$, is a Lie $A$-algebra structure on the tensor product $A \otimes_{\mathbb{K}} L$ with the bracket $[\,,\,]_\lambda$, such that

$$\varepsilon \otimes \mathrm{id}\colon A \otimes L \to \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism. (We usually abbreviate $\otimes_{\mathbb{K}}$ to $\otimes$.) See [Fi1], [Fi3].

EXAMPLE 1.2. If $A = \mathbb{K}[t]$, then a deformation of $L$ with base $A$ is the same as an algebraic 1-parameter deformation of $L$. More generally, if $A$ is the algebra of regular functions on an affine algebraic manifold $X$, then a deformation of $L$ with base $A$ is the same as an algebraic deformation of $L$ with base $X$.

Two deformations of a Lie algebra $L$ with the same base $A$ are called *equivalent* (or isomorphic) if there exists a Lie algebra isomorphism between the two copies of $A \otimes L$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$. A deformation with base $A$ is called *local* if the algebra $A$ is local, and it is called *infinitesimal* if, in addition to this, $\mathfrak{m}^2 = 0$.

DEFINITION 1.3. Let $A$ be a complete local algebra (completeness means that $A = \varprojlim_{n\to\infty} (A/\mathfrak{m}^n)$, where $\mathfrak{m}$ is the maximal ideal in $A$). A *formal deformation* of $L$ with base $A$ is a Lie $A$-algebra structure on the completed tensor product $A \,\hat{\otimes}\, L = \varprojlim_{n\to\infty} ((A/\mathfrak{m}^n) \otimes L)$ such that
$$\varepsilon \,\hat{\otimes}\, \mathrm{id}\colon A \,\hat{\otimes}\, L \to \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism (see [Fi3]) .

The above notion of equivalence is extended to formal deformations in an obvious way.

EXAMPLE 1.4. If $A = \mathbb{K}[[t]]$ then a formal deformation of $L$ with base $A$ is the same as a formal 1-parameter deformation of $L$. See [G], [NR].

Let $A'$ be another commutative algebra with identity over $\mathbb{K}$ with a fixed augmentation $\varepsilon' \colon A' \to \mathbb{K}$, and let $\varphi \colon A \to A'$ be an algebra homomorphism with $\varphi(1) = 1$ and $\varepsilon' \circ \varphi = \varepsilon$.

DEFINITION 1.5. If a deformation $\lambda$ of $L$ with base $(A, \mathfrak{m})$ is given, then the *push-out* $\varphi_* \lambda$ is the deformation of $L$ with base $(A', \mathfrak{m}' = \operatorname{Ker} \varepsilon')$, which is the Lie algebra structure

$$[a_1' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]' = a_1' a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2],$$

$$a_1', a_2' \in A', \ a_1, a_2 \in A, \ l_1, l_2 \in L,$$

on $A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L)$. Here $A'$ is regarded as an $A$-module with the structure $a'a = a'\varphi(a)$, and the operation $[\,,\,]$ in the right hand side of the formula refers to the Lie algebra structure $\lambda$ on $A \otimes L$.

The push-out of a formal deformation is defined in a similar way.

**1.2.** For completeness' sake, we recall the definition of Lie algebra cohomology (see [Fu]). We need only the case of cohomology with coefficients in the adjoint representation, and therefore we restrict our definition to this case.

Let

$$C^q(L; L) = \operatorname{Hom}(\Lambda^q L, L)$$

be the space of all skew-symmetric $q$-linear forms on a Lie algebra $L$ with values in $L$. Define the differential

$$\delta \colon C^q(L; L) \to C^{q+1}(L; L)$$

by the formula

$$(\delta \gamma)(l_1, \ldots, l_{q+1}) = \sum_{1 \leq s < t \leq q+1} (-1)^{s+t-1} \gamma([l_s, l_t], l_1, \ldots \hat{l}_s \ldots \hat{l}_t \ldots, l_{q+1})$$
$$+ \sum_{1 \leq u \leq q+1} [l_u, \gamma(l_1, \ldots \hat{l}_u \ldots, l_{q+1})],$$

where $\gamma \in C^q(L; L)$, $l_1, \ldots, l_{q+1} \in L$. It may be checked that $\delta^2 = 0$, and the cohomology of the complex $\{C^q(L; L), \delta\}$ is denoted by $H^q(L; L)$.

For $\alpha \in C^p(L; L)$, $\beta \in C^q(L; L)$, the Lie product $[\alpha, \beta]$ is defined be the formula

$$[\alpha, \beta](l_1, \ldots, l_{p+q-1})$$
$$= \sum_{1 \leq j_1 < \ldots < j_q \leq p+q-1} (-1)^{\Sigma_s (j_s - s)} \alpha(\beta(l_{j_1}, \ldots, l_{j_q}), l_1, \ldots \hat{l}_{j_1} \ldots \hat{l}_{j_q} \ldots, l_{p+q-1}) -$$

$$(-1)^{(p-1)(q-1)} \sum_{1 \leq k_1 < \ldots < k_p \leq p+q-1} (-1)^{\Sigma_t (k_t - t)} \beta(\alpha(l_{k_1}, \ldots, l_{k_p}), l_1, \ldots \hat{l}_{k_1} \ldots \hat{l}_{k_p} \ldots, l_{p+q-1}).$$

3

If one sets $\mathcal{C}^q = C^{q+1}(L; L)$, $\mathcal{H}^q = H^{q+1}(L; L)$, then this bracket operation (with the differential $\delta$) makes $\mathcal{C} = \bigoplus \mathcal{C}^q$ a differential graded Lie algebra (DGLA), and makes $\mathcal{H} = \bigoplus \mathcal{H}^q$ a graded Lie algebra.

**1.3.** Here is the fundamental example of an infinitesimal deformation of a Lie algebra. Consider a Lie algebra $L$ which satisfies the condition

$$\dim H^2(L; L) < \infty.$$

This is true, for example, if $\dim L < \infty$.

(There are some ways to weaken if not to completely avoid this condition. For example, if the Lie algebra $L$ is $\mathbb{Z}$-graded, $L = \bigoplus_{q \in \mathbb{Z}} L_{(q)}$, $[L_{(p)}, L_{(q)}] \subset L_{(p+q)}$, then $H^2(L; L)$ also becomes graded, $H^2(L; L) = \bigoplus_{q \in \mathbb{Z}} H^2_{(q)}(L; L)$, and the construction will be valid in a slightly modified form, if one supposes that $\dim H^2_{(q)}(L; L) < \infty$ for all $q$. See the details in 7.4 below.)

Consider the algebra

$$A = \mathbb{K} \oplus H^2(L; L)'$$

with the second summand being an ideal with zero multiplication ($'$ means the dual). Fix some homomorphism

$$\mu \colon H^2(L; L) \to C^2(L; L) = \operatorname{Hom}(\Lambda^2 L, L)$$

which takes a cohomology class into a cocycle representing this class. Define a Lie algebra structure on

$$A \otimes L = (\mathbb{K} \otimes L) \oplus (H^2(L; L)' \otimes L) = L \oplus \operatorname{Hom}(H^2(L; L), L)$$

by the formula

$$[(l_1, \varphi_1), (l_2, \varphi_2)] = ([l_1, l_2], \psi),$$

where

$$\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\varphi_1(\alpha), l_2] + [l_1, \varphi_2(\alpha)],$$

$$l_1, l_2 \in L, \ \varphi_1, \varphi_2 \in \operatorname{Hom}(H^2(L; L), L)), \alpha \in H^2(L; L).$$

(The Jacobi identity for this operation is implied by $\delta\mu(\alpha) = 0$.) This determines a deformation of $L$ with base $A$ which is clearly infinitesimal.

PROPOSITION 1.6. *Up to an isomorphism, this deformation does not depend on the choice of $\mu$.*

PROOF. Let

$$\mu' \colon H^2(L; L) \to C^2(L; L)$$

be another choice for $\mu$. Then there exists a homomorphism

$$\gamma \colon H^2(L; L) \to C^1(L; L) = \operatorname{Hom}(L, L)$$

such that $\mu'(\alpha) = \mu(\alpha) + \delta\gamma(\alpha)$ for all $\alpha \in H^2(L; L)$. Define a linear automorphism $\rho$ of the space $A \otimes L = L \oplus \mathrm{Hom}\,(H^2(L; L), L)$ by the formula

$$\rho(l, \varphi) = (l, \psi), \ \ \psi(\alpha) = \varphi(\alpha) + \gamma(\alpha)(l),$$

$$l \in L, \ \ \varphi \in \mathrm{Hom}\,(H^2(L; L), L), \ \ \alpha \in H^2(L; L).$$

The map $\rho$ is clearly an automorphism. The inverse of $\rho$ is given by replacing $\gamma$ with $-\gamma$ in the formula. To prove that $\rho$ is an isomorphism between the two Lie algebra structures, one needs to check that for any $l_1, l_2 \in L$, $\varphi_1, \varphi_2 \in \mathrm{Hom}\,(H^2(L; L), L)$, $\alpha \in H^2(L; L)$ one has

$$\mu(\alpha)(l_1, l_2) + [\varphi_1(\alpha), l_2] + [l_1, \varphi_2(\alpha)] + \gamma(\alpha)([l_1, l_1])$$
$$= \mu'(\alpha)(l_1, l_2) + [\varphi_1(\alpha) + \gamma(\alpha)(l_1), l_2] + [l_1, \varphi_2(\alpha) + \gamma(\alpha)(l_2)];$$

but this follows directly from the equality $\mu'(\alpha) = \mu(\alpha) + \delta\gamma(\alpha)$.

We will denote the infinitesimal deformation of $L$ constructed above by $\eta_L$.

**1.4.** The main property of $\eta_L$ is its (co-)universality in the class of infinitesimal deformations.

Let $\lambda$ be an infinitesimal deformation of the Lie algebra $L$ with the finite dimensional base $A$. Take $\xi \in \mathfrak{m}'$, or, equivalently, $\xi \in A'$ and $\xi(1) = 0$. For $l_1, l_2 \in L$ set

$$\alpha_{\lambda,\xi}(l_1, l_2) = (\xi \otimes \mathrm{id})[1 \otimes l_1, 1 \otimes l_2]_\lambda \in \mathbb{K} \otimes L = L.$$

LEMMA 1.7. *The cochain $\alpha_{\lambda,\xi} \in C^2(L; L)$ is a cocycle.*

PROOF. Let $l_1, l_2, l_3 \in L$. Since $[1 \otimes l_1, 1 \otimes l_2]_\lambda - 1 \otimes [l_1, l_2] \in \mathfrak{m} \otimes L$, we have

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_i m_i \otimes k_i,$$

where $m_i \in \mathfrak{m}, k_i \in L$. Hence

$$(\xi \otimes \mathrm{id})[1 \otimes l_1, 1 \otimes l_2]_\lambda = (\xi \otimes \mathrm{id})[1 \otimes [l_1, l_2], 1 \otimes l_3]_\lambda + (\xi \otimes \mathrm{id})\sum_i m_i[1 \otimes k_i, 1 \otimes l_3].$$

The first summand here is $\alpha_{\lambda,\xi}([l_1, l_2], l_3)$. For the second summand

$$m_i[1 \otimes k_i, 1 \otimes l_3] = m_i(1 \otimes [k_i, l_3] + h),$$

where $h \in \mathfrak{m} \otimes L$. Since $\mathfrak{m}^2 = 0$ we have $m_i h = 0$. Hence $m_i[1 \otimes k_i, 1 \otimes l_3] = m_i \otimes [k_i, l_3]$, and

$$(\xi \otimes \mathrm{id}) \sum_i m_i[1 \otimes k_i, 1 \otimes l_3] = \sum_i (\xi \otimes \mathrm{id})(m_i \otimes [k_i, l_3]) = \sum_i \xi(m_i)[k_i, l_3]$$
$$= \sum_i [\xi(m_i)k_i, l_3] = [(\xi \otimes \mathrm{id})\Big(\sum_i m_i \otimes k_i\Big), l_3]$$
$$= [(\xi \otimes \mathrm{id})([1 \otimes l_1, 1 \otimes l_2]_\lambda - 1 \otimes [l_1, l_2]), l_3]$$
$$= [(\xi \otimes \mathrm{id})[1 \otimes l_1, 1 \otimes l_2], l_3] = [\alpha_{\lambda,\xi}(l_1, l_2), l_3].$$

5

In the last step above we used that $\xi(1) = 0$. Thus

$$(\xi \otimes \mathrm{id})[[1 \otimes l_1, 1 \otimes l_2]_\lambda, 1 \otimes l_3]_\lambda = \alpha_{\lambda,\xi}([l_1, l_2], l_3) + [\alpha_{\lambda,\xi}(l_1, l_2), l_3],$$

and the Jacobi identity for $[\,,\,]_\lambda$ shows that $\alpha_{\lambda,\xi}$ is a cocycle.

PROPOSITION 1.8. *For any infinitesimal deformation $\lambda$ of the Lie algebra $L$ with a finite-dimensional base $A$ there exists a unique homomorphism $\varphi \colon \mathbb{K} \oplus H^2(L; L)' \to A$ such that $\lambda$ is equivalent to the push-out $\varphi_* \eta_L$.*

PROOF. For $\xi \in \mathfrak{m}'$ let $a_{\lambda,\xi} \in H^2(L; L)$ be the cohomology class of the cocycle $\alpha_{\lambda,\xi}$. The correspondences $\xi \mapsto \alpha_{\lambda,\xi}, \xi \mapsto a_{\lambda,\xi}$ define homomorphisms

$$\alpha_\lambda \colon \mathfrak{m}' \to C^2(L; L), \qquad \delta \circ \alpha_\lambda = 0,$$

$$a_\lambda \colon \mathfrak{m}' \to H^2(L; L).$$

We claim that
   (i) the deformation $\lambda$ is fully determined by $\alpha_\lambda$;
   (ii) the deformations $\lambda, \lambda'$ are equivalent if and only if $a_\lambda = a_{\lambda'}$;
   (iii) if $\varphi = \mathrm{id} \oplus a_\lambda' \colon \mathbb{K} \oplus H^2(L; L)' \to \mathbb{K} \oplus \mathfrak{m} = L$, then $\varphi_* \eta_L$ is equivalent to $\lambda$.
Since (ii) and (iii) obviously imply Proposition, it remains to prove (i)–(iii). The statement (i) is obvious. To prove (ii) notice that an $A$-automorphism $\rho \colon A \otimes L \to A \otimes L$, that is

$$L \oplus (\mathfrak{m} \otimes L) \to L \oplus (\mathfrak{m} \otimes L),$$

whose $L \to L$ part is the identity (this is the condition of compatibility with $\varepsilon \otimes \mathrm{id}$), is fully determined by its

$$L \to \mathfrak{m} \otimes L$$

part, which we denote by $b_\rho$, and the latter may be chosen arbitrarily. This is an element of

$$\mathrm{Hom}\,(L, \mathfrak{m} \otimes L) = \mathfrak{m} \otimes \mathrm{Hom}\,(L, L) = \mathfrak{m} \otimes C^1(L; L) = \mathrm{Hom}\,(\mathfrak{m}', C^1(L; L)).$$

It is easy to check that $\rho$ establishes an isomorphism between the Lie algebra structures $\lambda$ and $\lambda'$ if and only if

$$\alpha_{\lambda'} - \alpha_\lambda = \delta \circ b_\rho,$$

which proves (ii). Finally, it follows from the definitions that

$$\alpha_{\varphi_* \eta_L} = \mu \circ a_\lambda,$$

which implies that $a_{\varphi_* \eta_L} = a_\lambda$, and hence $\varphi_* \eta_L$ and $\lambda$ are isomorphic as was stated in (iii).

REMARK 1.9. Technically, the mapping $a_\lambda \colon \mathfrak{m}' \to H^2(L; L)$ constructed in the proof will be more important for us than the map $\varphi = \mathrm{id} \oplus a_\lambda'$.

Let $A$ be a local algebra with $\dim(A/\mathfrak{m}^2) < \infty$. Obviously, $A/\mathfrak{m}^2$ is local with the maximal ideal $\mathfrak{m}/\mathfrak{m}^2$, and $(\mathfrak{m}/\mathfrak{m}^2)^2 = 0$. Recall that the dual space $(\mathfrak{m}/\mathfrak{m}^2)'$ is called the *tangent space* of $A$; we denote it by $TA$.

DEFINITION 1.10. Let $\lambda$ be a deformation of $L$ with base $A$. Then the mapping

$$a_{\pi_*\lambda}: TA = (\mathfrak{m}/\mathfrak{m}^2)' \to H^2(L; L),$$

where $\pi$ is the projection $A \to A/\mathfrak{m}^2$, is called the *differential* of $\lambda$ and is denoted by $d\lambda$.

The differential of a formal deformation is defined in a similar way.

It is clear from the construction that equivalent deformations or formal deformations have equal differentials.

**1.5.** It is not possible to construct a local or formal deformation of a Lie algebra with a similar universality property in the class of local or formal deformations. But it becomes possible for an appropriate weakening of this property.

DEFINITION 1.11. A formal deformation $\eta$ of a Lie algebra $L$ with base $B$ is called *miniversal* if

(i) for any formal deformation $\lambda$ of $L$ with any (local) base $A$ there exists a homomorphism $f: B \to A$ such that the deformation $\lambda$ is equivalent to $f_*\eta$;

(ii) in the notations of (i), if $A$ satisfies the condition $\mathfrak{m}^2 = 0$, then $f$ is unique (see [Fi1]).

If $\eta$ satisfies only the condition (i), then it is called *versal*.

Our goal is to construct a miniversal formal deformation of a given Lie algebra.

## 2. Harrison cohomology.

**2.1.** We will need a special cohomology theory for commutative algebras introduced in 1961 by D. K. Harrison [Harr]. The following general definition is contained in the article [B].

Let $A$ be a commutative $\mathbb{K}$-algebra. Consider the standard Hochschild complex $\{C_q(A), \partial\}$ for $A$. Here $C_q(A)$ is the $A$-module $A^{q+1} = A \otimes \ldots \otimes A$ ($q+1$ factors), $A$ operates on the last factor, and the differential $\partial: C_q(A) \to C_{q-1}(A)$ is defined by the formula

$$\partial[a_1, \ldots, a_q] = a_1[a_2, \ldots, a_q] + \sum_{i=1}^{q-1} (-1)^q [a_1, \ldots, a_i a_{i+1}, \ldots, a_q]$$
$$+ (-1)^q a_q[a_1, \ldots, a_{q-1}],$$

where $b_0[b_1, \ldots, b_n]$ means $b_0 \otimes b_1 \otimes \ldots \otimes b_n \in C_n(A)$. A permutation from $S(q)$ is called a $(p, q-p)$-shuffle if the inverse permutation $(j_1, \ldots, j_q)$ satisfies the conditions $j_1 < \ldots < j_p$, $j_{p+1} < \ldots < j_q$. Let $\mathrm{Sh}\,(p, q-p) \subset S(q)$ be the set of all $(p, q-p)$-shuffles. For $a_1, \ldots, a_q \in A$ and $0 < p < q$ set

$$s_p(a_1, \ldots, a_q) = \sum_{(i_1, \ldots, i_q) \in \mathrm{Sh}(p, q-p)} \mathrm{sgn}(i_1, \ldots, i_q)[a_{i_1}, \ldots, a_{i_q}] \in C_q(A).$$

Let $\mathrm{Sh}_q(A)$ be the $A$-submodule of $C_q(A)$ generated by the chains $s_p(a_1, \ldots, a_q)$ for all $a_1, \ldots, a_q \in A$, $0 < p < q$. It may be checked (see [B], Proposition 2.2) that $\partial(\mathrm{Sh}_q(A)) \subset$

$\mathrm{Sh}_{q-1}(A)$, which yields a complex $\mathrm{Ch}\,(A) = \{\mathrm{Ch}_q(A) = C_q(A)/\mathrm{Sh}_q(A), \partial\}$. This is the *Harrison complex.*

DEFINITION 2.1. For an $A$-module $M$ we set

$$H_q^{\mathrm{Harr}}(A; M) = H_q(\mathrm{Ch}\,(A) \otimes M),$$
$$H_{\mathrm{Harr}}^q(A; M) = H^q(\mathrm{Hom}\,(\mathrm{Ch}\,(A), M);$$

these are *Harrison homology and cohomology of $A$ with coefficients in $M$.* (For the relations between Harrison and Hochschild homology and cohomology see [B].)

We will need the following standard fact, which follows directly from the definition.

PROPOSITION 2.2. *Let $A$ be a local commutative $\mathbb{K}$- algebra with the maximal ideal $\mathfrak{m}$, and let $M$ be an $A$-module with $\mathfrak{m}M = 0$. Then we have the canonical isomorphisms*

$$H_q^{\mathrm{Harr}}(A; M) \cong H_q^{\mathrm{Harr}}(A; \mathbb{K}) \otimes M, \ \ H_{\mathrm{Harr}}^q(A; M) \cong H_{\mathrm{Harr}}^q(A; \mathbb{K}) \otimes M.$$

**2.2.** We will need only 1- and 2-dimensional Harrison cohomology. Here is their direct description (belonging to Harrison [Harr]). Let $A$ and $M$ be as above. Consider the complex

$$0 \to \mathrm{Ch}^1 \xrightarrow{d_1} \mathrm{Ch}^2 \xrightarrow{d_2} C^3,$$

where

$$\mathrm{Ch}^1 = \mathrm{Hom}\,(A, M), \ \mathrm{Ch}^2 = \mathrm{Hom}\,(S^2 A, M), \ C^3 = \mathrm{Hom}\,(A \otimes A \otimes A, M),$$

$$d_1\psi(a, b) = a\psi(b) - \psi(ab) + b\psi(a), \ d_2\varphi(a, b, c) = a\varphi(b, c) - \varphi(ab, c) + \varphi(a, bc) - c\varphi(a, b),$$

$$m \in M, \ a, b, c \in A, \ \psi \in \mathrm{Ch}^1, \ \varphi \in \mathrm{Ch}^2.$$

PROPOSITION 2.3. (i) $H_{\mathrm{Harr}}^1(A; M)$ *is the space of derivations $A \to M$.* (ii) *Elements of $H_{\mathrm{Harr}}^2(A; M)$ correspond bijectively to isomorphism classes of extensions $0 \to M \to B \to A \to 0$ of the algebra $A$ by means of $M$.*

PROOF. Part (i) is obvious. To prove (ii), consider an extension $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ and fix a section $q: A \to B$ of $p$. Then $b \mapsto (p(b), i^{-1}(b - q \circ p(b)))$ is an isomorphism $B \to A \oplus M$. Let $(a, m)_q \in B$ be the inverse image of $(a, m) \in A \oplus M$ with respect to this isomorphism. For $a_1, a_2 \in A$ set $\varphi_q(a_1, a_2) = i^{-1}((a_1, 0)_q(a_2, 0)_q - (a_1 a_2, 0)_q) \in M$. Then the multiplication in $B$ is $(a_1, m_1)_q(a_2, m_2)_q = (a_1 a_2, a_1 m_2 + a_2 m_1 + \varphi_q(a_1, a_2))_q$, so it is determined by $\varphi_q$. Furthermore, the associativity of the algebra $B$ implies that $\varphi_q \in \mathrm{Ch}^2$ is a cocycle. For any other section $q': A \to B$ one has $i^{-1} \circ (q' - q) \in \mathrm{Ch}^1$, and it is easy to check that $\varphi_{q'} = \varphi_q + d_1(i^{-1} \circ (q' - q))$. This implies (ii).

COROLLARY 2.4. *If $A$ is a local algebra with the maximal ideal $\mathfrak{m}$, then $H_{\mathrm{Harr}}^1(A; \mathbb{K}) = (\mathfrak{m}/\mathfrak{m}^2)' = TA$.*

PROOF Let $\varphi: A \to \mathbb{K}$ be a derivation. If $a \in \mathfrak{m}^2$, that is $a = a_1 a_2$, $a_1, a_2 \in \mathfrak{m}$, then $\varphi(a) = \varphi(a_1 a_2) = a_1\varphi(a_2) + a_2\varphi(a_1) = 0$ (since $\mathfrak{m}\mathbb{K} = 0$). Furthermore, $\varphi(1) = \varphi(1 \cdot 1) = 1\varphi(1) + 1\varphi(1) = 2\varphi(1)$, hence $\varphi(1) = 0$. On the other hand, any homomorphism $\varphi: A \to \mathbb{K}$

such that $\varphi(\mathfrak{m}^2) = 0, \varphi(1) = 0$, is a derivation. Thus the space of derivations $A \to \mathbb{K}$ is $(\mathfrak{m}/\mathfrak{m}^2)'$.

PROPOSITION 2.5. *Let $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ be an extension of an algebra $A$ . (i) If $A$ has an identity, then so does $B$. (ii) If $A$ is local with the maximal ideal $\mathfrak{m}$, then $B$ is local with the maximal ideal $p^{-1}(\mathfrak{m})$.*

PROOF. (i) We use the notations of the previous proof. Fix a section $q \colon A \to B$ of $p$. Then we get a cocycle $\varphi = \varphi_q \in \mathrm{Ch}^2$. For any $a \in A$

$$d_2\varphi(1, 1, a) = \varphi(1, a) - \varphi(1, a) + \varphi(1, a) - a\varphi(1, 1) = 0,$$

which shows that $\varphi(1, a) = a\varphi(1, 1)$. Consider an arbitrary $\psi \in \mathrm{Ch}^1$ with $\psi(1) = \varphi(1, 1)$. Let $\varphi' = \varphi - d_1\psi$. Then for any $a \in A$

$$\begin{aligned}
\varphi'(1, a) &= \varphi(1, a) - d_1\psi(1, a) \\
&= \varphi(1, a) - \psi(a) + \psi(a) - a\psi(1) \\
&= \varphi(1, a) - a\varphi(1, 1) = 0.
\end{aligned}$$

According to the previous proof, $\varphi' = \varphi_{q'}$ for some section $q' \colon A \to B$, and one has

$$(1, 0)_{q'}(a, m)_{q'} = (a, m + \varphi_{q'}(1, a))_{q'} = (a, m)_{q'}.$$

Hence, $(1, 0)_{q'} \in B$ is the unit element.

(ii) Let $\mathfrak{n} \subset B$ be an ideal, and let $\mathfrak{n} \not\subset p^{-1}(\mathfrak{m})$. Then there is some $b \in \mathfrak{n}$ such that $p(b) = 1$. Choose a section $q \colon A \to B$ with $q(1) = b$. Then $b = (1, 0)_q$. For any $(a, m)_q \in B$ one has

$$(a, m)_q = (1, 0)_q(a, m - \varphi_q(1, a)) \in \mathfrak{n},$$

and hence $\mathfrak{n} = B$.

**2.3.** The relationship between the second Harrison cohomology of a finite-dimensional local commutative algebra $A$ and extensions of $A$ may be also described in terms of one remarkable extension. This is the extension

$$0 \to H^2_{\mathrm{Harr}}(A; \mathbb{K})' \to C \to A \to 0, \tag{1}$$

where the operation of $A$ on $H^2_{\mathrm{Harr}}(A; \mathbb{K})'$ is induced by the operation of $A$ on $\mathbb{K}$, and the cocycle

$$f_A : S^2 A \to H^2_{\mathrm{Harr}}(A; \mathbb{K})'$$

is defined as the dual of a homomorphism

$$\mu : H^2_{\mathrm{Harr}}(A; \mathbb{K}) \to \mathrm{Ch}^2(A; \mathbb{K}) = (S^2 A)',$$

which takes a cohomology class to a cocycle from this class. This extension does not depend, up to an isomorphism, on the choice of $\mu$ (compare Proposition 1.6) and possesses the following partial (co-)universality property.

9

PROPOSITION 2.6. *Let $M$ be an $A$-module with $\mathfrak{m}M = 0$. Then the extension (1) admits a unique homomorphism into an arbitrary extension $0 \to M \to B \to A \to 0$ of $A$.*

PROOF. The extension $0 \to M \to B \to A \to 0$ corresponds to some element of $H^2_{\mathrm{Harr}}(A; M) = H^2_{\mathrm{Harr}}(A; \mathbb{K}) \otimes M$ (see Proposition 2.2). The latter defines a mapping $H^2_{\mathrm{Harr}}(A; \mathbb{K})' \to M$, which implies, in turn, a mapping $C \to B$. The resulting diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2_{\mathrm{Harr}}(A; \mathbb{K})' & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

is an extension homomorphism. Its uniqueness is obvious.

**2.4.** $H^1_{\mathrm{Harr}}(A; M)$ is also interpreted as the set of automorphisms of any given extension $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ of $A$. An automorphism is an algebra automorphism $f\colon B \to B$ such that $f \circ i = i$ and $p \circ f = p$. In previous notations (see Proof of Proposition 2.3), $f(a, m)_q = (f_1(a, m), f_2(a, m))_q$. The condition $p \circ f = p$ means that $f_1(a, m) = a$. The condition $f \circ i = f$ means that $f_2(0, m) = m$, which implies that $f_2(a, m) = f_2(a, 0) + f_2(0, m) = m + \psi(a)$ (where $\psi(a) = f_2(a, 0)$). The multiplicativity of $f$ implies successively

$$f((a_1, 0)_q, (a_2, 0)_q) = f(a_1, 0)_q f(a_2, 0)_q,$$
$$f(a_1 a_2, \varphi_q(a_1, a_2))_q = (a_1, \psi(a_1))_q (a_2, \psi(a_2))_q,$$
$$(a_1 a_2, \varphi_q(a_1, a_2) + \psi(a_1 a_2))_q = (a_1 a_2, \varphi_q(a_1, a_2) + a_1 \psi(a_2) + a_2 \psi(a_1)),$$
$$\psi(a_1 a_2) = a_1 \psi(a_2) + a_2 \psi(a_1),$$

that is $d_1 \psi = 0$. Conversely, any $\psi\colon A \to M$ with $d_1 \psi = 0$ determines an algebra automorphism $f\colon B \to B, (a, m)_q \mapsto (a, m + \psi(a))_q$ with the required properties.

Notice that $f(1, 0)_q = (1, \psi(1))_q = (1, 0)_q$, because $\psi(1) = 0$ for any derivation $\psi$. Hence $f$ takes the unit element of $B$ into the unit element of $B$ (cf. Proposition 2.4).

**2.5.** In Section 4 we will use the following result due to Harrison.

PROPOSITION 2.7. ([Harr], Theorems 11 and 18). *Let $A = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial algebra, and let $\mathfrak{m}$ be the ideal of polynomials without constant terms. If an ideal $I$ of $A$ is contained in $\mathfrak{m}^2$, then*

$$H^2_{\mathrm{Harr}}(A/I; \mathbb{K}) \cong (I/(\mathfrak{m} \cdot I))'.$$

Harrison's work contains also an explicit construction of the above homomorphism, which implies the following description of the canonical extension

$$0 \to H^2_{\mathrm{Harr}}(B; \mathbb{K})' \to C \to B \to 0$$

of $B = A/I$ (see 2.3).

PROPOSITION 2.8. *If $A$, $\mathfrak{m}$, and $I$ are as in Proposition 2.7, then the preceding extension for $B = A/I$ is*

$$0 \to I/(\mathfrak{m} \cdot I) \xrightarrow{i} A/(\mathfrak{m} \cdot I) \xrightarrow{p} A/I \to 0,$$

*where $i$ and $p$ are induced by the inclusions $I \to A$ and $\mathfrak{m} \cdot I \to I$.*

# 3. Obstructions to extending deformations

**3.1.** Let $\lambda$ be a deformation of a Lie algebra $L$ with a finite-dimensional local base $A$, and let $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ be a 1-dimensional extension of $A$, corresponding to a cohomology class $f \in H^2_{\mathrm{Harr}}(A; \mathbb{K})$.

Let $I = i \otimes \mathrm{id} \colon L = \mathbb{K} \otimes L \to B \otimes L$ and $P = p \otimes \mathrm{id} \colon B \otimes L \to A \otimes L$. Let also $E = \hat{\varepsilon} \otimes \mathrm{id} \colon B \otimes L \to \mathbb{K} \otimes L = L$, where $\hat{\varepsilon}$ is the augmentation of $B$. The Lie algebra structure $[\,,\,]_\lambda$ in $A \otimes L$ can be lifted to a $B$-bilinear operation $\{\,,\,\} \colon \Lambda^2 B \to B$ such that

(i) $P\{l_1, l_2\} = [P(l_1), P(l_2)]_\lambda$ for any $l_1, l_2 \in B \otimes L$,
(ii) $\{I(l), l_1\} = I[l, E(l_1)]$ for any $l \in L, l_1 \in B \otimes L$.

The operation $\{\,,\,\}$ partially satisfies the Jacobi identity, that is

$$\varphi(l_1, l_2, l_3) := \{l_1, \{l_2, l_3\}\} + \{l_2, \{l_3, l_1\}\} + \{l_3, \{l_1, l_2\}\} \in \mathrm{Ker}\, P.$$

Remark that $\varphi$ is multilinear and skew-symmetric, and $\varphi(l_1, l_2, l_3) = 0$ if $l_1 \in \mathrm{Ker}\, E$. (Indeed, if $l_1 = ml'_1$, where $m \in \hat{\mathfrak{m}} = \mathrm{Ker}\, \hat{\varepsilon}$, then $\varphi(l_1, l_2, l_3) = \varphi(ml'_1, l_2, l_3) = m\varphi(l'_1, l_2, l_3) = 0$.) Hence $\varphi$ determines a multilinear form

$$\bar{\varphi} \colon \Lambda^3 L = \Lambda^3((B \otimes L)/\mathrm{Ker}\, E) \to \mathrm{Ker}\, P = L,$$

that is an element $\bar{\varphi}$ of $C^3(L; L)$. It is easy to check that $\delta\bar{\varphi} = 0$.

Let $\{\,,\,\}'$ be another $B$-bilinear operation $\Lambda^2 B \to B$ satisfying the conditions (i), (ii) above. Then $\{l_1, l_2\}' - \{l_1, l_2\} \in \mathrm{Ker}\, P$ for any $l_1, l_2 \in B \otimes L$, and if $l_1 \in \mathrm{Ker}\, E$ then $\{l_1, l_2\}' - \{l_1, l_2\} = 0$ (as above, if $l_1 = ml'_1, m \in \hat{\mathfrak{m}}$, then $\{l_1, l_2\}' - \{l_1, l_2\} = \{ml'_1, l_2\}' - \{ml'_1, l_2\} = m(\{l'_1, l_2\}' - \{l_1, l_2\}) = 0$.) Hence the difference $\{\,,\,\}' - \{\,,\,\}$ determines a form $\psi \colon \Lambda^2 L = \Lambda^2((B \otimes L)/\mathrm{Ker}\, E) \to \mathrm{Ker}\, P = L$, that is determines a cochain $\psi \in C^2(L; L)$. Moreover, an arbitrary cochain $\psi \in C^2(L; L)$ may be obtained as $\{\,,\,\}' - \{\,,\,\}$ with an appropriate $\{\,,\,\}'$.

Using the cocycle $f_A$, it is easy to check that if $\bar{\varphi}, \bar{\varphi}' \in C^3(L; L)$ are the cochains corresponding to $\{\,,\,\}, \{\,,\,\}'$ in the sense of the construction above, then

$$\bar{\varphi}' - \bar{\varphi} = \delta\psi.$$

Let $\mathcal{O}_\lambda(f) \in H^3(L; L)$ be the cohomology class of the cochain $\bar{\varphi}$. It is obvious that

$$\mathcal{O}_\lambda \colon H^2_{\mathrm{Harr}}(A; \mathbb{K}) \to H^3(L; L), \ f \mapsto \mathcal{O}_\lambda(f)$$

is a linear map.

We can summarize the argumentation above in the following

PROPOSITION 3.1. *The deformation $\lambda$ with base $A$ can be extended to a deformation of $L$ with base $B$ if and only if $\mathcal{O}_\lambda(f) = 0$.*

The cohomology class $\mathcal{O}_\lambda(f)$ is called the *obstruction* to the extension of the deformation $\lambda$ from $A$ to $B$.

**3.2.** Suppose now that $\mathcal{O}_\lambda(f) = 0$, that is the deformation $\lambda$ is extendible to a deformation with base $B$. We are going to study the set of all possible extensions.

Let $\mu, \mu'$ be deformations of $L$ with base $B$ such that $p_*\mu = p_*\mu' = \lambda$. Then, according to 3.1, the difference $[\,,\,]_{\mu'} - [\,,\,]_\mu$ determines and is determined by a certain cochain $\psi \in C^2(L; L)$. Since $[\,,\,]_{\mu'}$ and $[\,,\,]_\mu$ both satisfy the Jacobi identity, $\delta\psi = 0$. Moreover, it is easy to check that if we replace any of the structures $[\,,\,]_\mu, [\,,\,]_{\mu'}$ with an equivalent one (see 1.1), then the cocycle $\psi$ will be replaced by a cohomologous cocycle. Thus the difference between two isomorphism classes of deformations $\mu$ of $L$ with base $B$ such that $p_*\mu = \lambda$ is an arbitrary element of $H^2(L; L)$. In other words, $H^2(L; L)$ operates transitively on the set of these equivalence classes.

On the other hand, the group of automorphisms of the extension $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ also operates on the set of equivalence classes of deformations $\mu$. According to 2.5, this group is $H^1_{\mathrm{Harr}}(A; \mathbb{K})$, and according to Corollary 2.4, $H^1_{\mathrm{Harr}}(A; \mathbb{K}) = (\mathfrak{m}/\mathfrak{m}^2)' = TA$.

PROPOSITION 3.2. *These two operations are related to each other by the differential* $d\lambda\colon TA \to H^2(L; L)$ *(see Definition 1.10). In other words, if* $r\colon B \to B$ *determines an automorphism of the extension* $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ *which corresponds to an element* $h \in H^1_{\mathrm{Harr}}(A; \mathbb{K}) = TA$, *then for any deformation* $\mu$ *of* $L$ *with base* $B$ *such that* $p_*\mu = \lambda$, *the difference between* $[\,,\,]_{r_*\mu}$ *and* $[\,,\,]_\mu$ *is a cocycle of the cohomology class* $d\lambda(h)$.

PROOF is obvious.

COROLLARY 3.3. *Suppose that the differential* $d\lambda\colon TA \to H^2(L; L)$ *is onto. Then the group of automorphisms of the extensions* $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ *operates transitively on the set of equivalence classes of deformations* $\mu$ *of* $L$ *with base* $B$ *such that* $p_*\mu = \lambda$. *In other words,* $\mu$ *is unique up to an isomorphism and an automorphism of the extension* $0 \to \mathbb{K} \to B \to A \to 0$.

**3.3.** The results of 3.1 and 3.2 may be generalized from the case of extension $0 \to \mathbb{K} \to B \to A \to 0$ to a more general case of extensions $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$, where $M$ is a finite-dimensional $A$-module satisfying the condition $\mathfrak{m}M = 0$. The construction of 3.1 applied to a deformation $\lambda$ of $L$ with base $A$ yields an element of $H^3(L; M \otimes L) = M \otimes H^3(L; L)$.

The same element may be obtained from the previous construction in a more direct way. Let $h \in M'$. We set $B_h = (B \oplus \mathbb{K})/\mathrm{Im}\,(i \oplus h)$ (that is $B_h = B/i(\mathrm{Ker}\,h)$ if $h \neq 0$, and $B_0 = A \oplus \mathbb{K}$). There is an obvious extension $0 \to \mathbb{K} \to B_h \to A \to 0$; let $f_h \in H^2_{\mathrm{Harr}}(A; \mathbb{K})$ be the corresponding cohomology class. The formula $h \mapsto \mathcal{O}_\lambda(f_h)$ defines an element of $\mathrm{Hom}\,(M', H^3(L; L)) = M \otimes H^3(L; L)$ which coincides with the obstruction constructed above.

PROPOSITION 3.4. *A deformation* $\mu$ *of* $L$ *with base* $B$ *such that* $p_*\mu = \lambda$ *exists if and only if the element of* $M \otimes H^3(L; L)$ *constructed above is equal to* 0. *If* $d\lambda\colon TA \to H^3(L; L)$ *is onto then the deformation* $\mu$, *if it exists, is unique up to an isomorphism and an automorphism of the extension* $0 \to M \to B \to A \to 0$.

PROOF is as above (see 3.1).

## 4. Construction of a miniversal deformation

**4.1.** Suppose that $\dim H^2(L; L) < \infty$.

Let $C_0 = \mathbb{K}$, $C_1 = \mathbb{K} \oplus H^2(L; L)'$, and let

$$0 \to H^2(L; L)' \xrightarrow{i_1} C_1 \xrightarrow{p'_1} \mathbb{K} \to 0$$

be the canonical splitting extension. The deformation $\eta_L$ of $L$ with base $C_1$ constructed in 1.3 will be denoted here by $\eta_1$. Suppose that for some $k \geq 1$ we have already constructed a finite-dimensional commutative algebra $C_k$ and a deformation $\eta_k$ of $L$ with base $C_k$. Consider the extension

$$0 \to H^2_{\mathrm{Harr}}(C_k; \mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}'_{k+1}} C_k \to 0 \tag{2}$$

constructed in 2.3 using the cocycle $f_{C_k}$ (the notation was different there). According to 3.3, we obtain the obstruction

$$\mathcal{O}_{\eta_k}(f_{C_k}) \in H^2_{\mathrm{Harr}}(C_k, \mathbb{K})' \otimes H^3(L; L)$$

to the extension of $\eta_k$. This gives us a map

$$\omega_k \colon H^2_{\mathrm{Harr}}(C_k, \mathbb{K}) \to H^3(L; L).$$

Set

$$C_{k+1} = \bar{C}_{k+1} / \bar{i}_{k+1} \circ \omega'_k(H^3(L; L)').$$

Obviously, the extension (2) factorizes to an extension

$$0 \to (\mathrm{Ker}\,\omega_k)' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p'_{k+1}} C_k \to 0. \tag{3}$$

Notice that all the algebras $C_k$ are local. Since $C_k$ is finite-dimensional, the cohomology $H^2_{\mathrm{Harr}}(C_k; \mathbb{K})$ is also finite-dimensional, and hence $C_{k+1}$ is finite-dimensional.

PROPOSITION 4.1. *The deformation $\eta_k$ admits an extension to a deformation with base $C_{k+1}$, and this extension is unique up to an isomorphism and an automorphism of an extension* (3).

PROOF. According to Proposition 3.4, the obstruction to the extension of the deformation $\eta_k$ of $L$ from $C_k$ to $C_{k+1}$ is a homomorphism $\mathrm{Ker}\,\omega_k \to H^3(L; L)$, and it is easy to show that it is precisely the restriction of $\omega_k$. Hence it is equal to 0. The uniqueness of the extension is stated explicitly in Proposition 3.4.

We choose an extended deformation and denote it by $\eta_{k+1}$.

The induction yields a sequence of finite-dimensional algebras

$$\mathbb{K} \xleftarrow{p'_1} C_1 \xleftarrow{p'_2} \ldots \xleftarrow{p'_k} C_k \xleftarrow{p'_{k+1}} C_{k+1} \xleftarrow{p'_{k+2}} \ldots,$$

and a sequence of deformations $\eta_k$ of $L$ such that $(p'_{k+1})_*\eta_{k+1} = \eta_k$.

Taking the projective limit, we obtain a formal deformation $\eta$ of $L$ with base $C = \varprojlim_{k\to\infty} C_k$. In Theorem 4.5 below we will show that $\eta$ is a miniversal deformation of $L$.

**4.2.** Denote the space $H^2(L; L)$ briefly by $\boldsymbol{H}$. Below we assume that $\dim \boldsymbol{H} < \infty$. Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{K}[[\boldsymbol{H}']]$.

PROPOSITION 4.2. $C_k = \mathbb{K}[[\boldsymbol{H}']]/I_k$ where

$$\mathfrak{m}^2 = I_1 \supset I_2 \supset \ldots, \ I_k \supset \mathfrak{m}^{k+1}.$$

PROOF. By construction,

$$C_1 = \mathbb{K} \oplus \boldsymbol{H}' = \mathbb{K}[[\boldsymbol{H}']]/\mathfrak{m}^2.$$

Suppose that we already know that

$$C_k = \mathbb{K}[[\boldsymbol{H}']]/I_k, \ \mathfrak{m}^2 \supset I_k \supset \mathfrak{m}^{k+1}.$$

Then, according to Proposition 2.8,

$$\bar{C}_{k+1} = \mathbb{K}[[\boldsymbol{H}']]/(\mathfrak{m} \cdot I_k),$$

and by construction $C_{k+1}$ is the quotient of $\bar{C}_{k+1}$ over an ideal contained in $I_k/(\mathfrak{m} \cdot I_k) \subset \mathfrak{m}^2/(\mathfrak{m} \cdot I_k)$. Hence

$$C_{k+1} = \mathbb{K}[[\boldsymbol{H}']]/I_{k+1}, \ \text{where} \ \mathfrak{m}^2 \supset I_{k+1} \supset \mathfrak{m} \cdot I_k \supset \mathfrak{m}^{k+2}.$$

This completes the proof.

COROLLARY 4.3. For $k \geq 2$ the projection $p'_k : C_k \to C_{k-1}$ implies an isomorphism $TC_k \to TC_{k-1}$. In particular, the space $TC_k$ does not depend on $k$; $TC_k = TC_1 = H^2(L; L)$. More precisely, for any $k \geq 1$ the differential $d\eta_k : TC_k \to H^2(L; L)$ is an isomorphism.

PROPOSITION 4.4. $C = \mathbb{K}[[\boldsymbol{H}']]/I$, where $I$ is an ideal contained in $\mathfrak{m}^2$. Note that since $\mathbb{K}[[\boldsymbol{H}']]$ is Noetherian, then $I$ is finitely generated.

PROOF. By construction, $C = \varprojlim_{k\to\infty} C_k$ (see 4.1). Proposition 4.2 gives an epimorphism

$$\varprojlim_{k\to\infty} \mathbb{K}[[\boldsymbol{H}']]/\mathfrak{m}^{k+1} \to \varprojlim_{k\to\infty} C_k,$$

that is

$$\mathbb{K}[[\boldsymbol{H}']] \to C,$$

and

$$C = \mathbb{K}[[\boldsymbol{H}']]/I, \ \text{where} \ I = \cap I_k = \varprojlim I_k.$$

**4.3** THEOREM 4.5. *If* $\dim H^2(L; L) < \infty$, *then the formal deformation* $\eta$ *is a miniversal formal deformation of* $L$.

PROOF. Since $TC_k = H^2(L; L)$ and $d\eta_k = \mathrm{id}$, then $TC = H^2(L; L)$ and $d\eta = \mathrm{id}$. Let $A$ be a complete local algebra with the maximal ideal $\mathfrak{m}$, and let $\lambda$ be a deformation of $L$ with base $A$. We put $A_0 = A/\mathfrak{m} = \mathbb{K}$ and $A_1 = A/\mathfrak{m}^2 = \mathbb{K} \oplus (TA)'$. Then we fix a sequence of 1-dimensional extensions

$$0 \to \mathbb{K} \xrightarrow{j_{k+1}} A_{k+1} \xrightarrow{q_{k+1}} A_k \to 0, \ k \geq 1$$

such that $A = \varprojlim\limits_{k\to\infty} A_k$. Let $Q_k: A \to A_k$ be the projection; we suppose that $Q_1$ is the natural projection $A \to A/\mathfrak{m}^2$. Let $\lambda_k = (Q_k)_*\lambda$; it is a deformation of $L$ with base $A_k$. Obviously, $\lambda_k = (q_{k+1})_*\lambda_{k+1}$. We will construct inductively homomorphisms $\varphi_j: C_j \to A_j$, $j = 1, 2, \ldots$ compatible with the projections $C_{j+1} \to C_j$, $A_{j+1} \to A_j$ and such that $(\varphi_j)_*\eta_j = \lambda_j$.

Define $\varphi_1: C_1 \to A_1$ as $\mathrm{id} \oplus (d\lambda)': \mathbb{K} \oplus H^2(L; L)' \to \mathbb{K} \oplus (TA)'$; by definition of the differential, $(\varphi_1)_*\eta_1 = \lambda_1$. Suppose that $\varphi_k: C_k \to A_k$ with $(\varphi_k)_*\eta_k = \lambda_k$ has been already constructed. The homomorphism $\varphi_k^*: H^2_{\mathrm{Harr}}(A_k; \mathbb{K}) \to H^2_{\mathrm{Harr}}(C_k; \mathbb{K})$ induced by $\varphi_k$ takes the class of extension $0 \to \mathbb{K} \to A_{k+1} \to A_k \to 0$ into the class of some extension $0 \to \mathbb{K} \to B \to C_k \to 0$, and we have a homomorphism

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi_k} & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & A_{k+1} & \longrightarrow & A_k & \longrightarrow & 0
\end{array}
$$

Obviously, there exists a deformation $\xi$ of $L$ with base $B$ which extends $\eta_k$ (because the deformations $\lambda_k$ and $\eta_k$ have the same obstruction to extension) and such that $\psi_*\xi = \lambda_{k+1}$ (extensions of $\lambda_k$ and $\eta_k$ are both parameterized by $H^2(L; L)$).

According to Proposition 2.6, there exists a homomorphism

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2_{\mathrm{Harr}}(C_k; \mathbb{K})' & \xrightarrow{\bar{i}_{k+1}} & \bar{C}_{k+1} & \xrightarrow{\bar{p}'_{k+1}} & C_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle\bar{\chi}} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0
\end{array}
$$

and since the deformation $\eta_k$ is extended to $B$, it follows that the composition

$$r \circ \omega_k': H^3(L; L)' \to \mathbb{K}$$

is zero. Hence the last diagram may be factorized to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathrm{Ker}\,\omega_k)' & \longrightarrow & C_{k+1} & \longrightarrow & C_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0
\end{array}
$$

15

Since $d\eta_k\colon TC_k \to H^2(L;L)$ is an epimorphism (see 1.4.1), the two deformations $\chi_*\eta_{k+1}$ and $\xi$ are related by some automorphism $f\colon B \to B$ of the extension $0 \to \mathbb{K} \to B \to C_k \to 0$. It remains to set $\varphi_{k+1} = \psi \circ f \circ \chi\colon C_{k+1} \to A_{k+1}$; indeed, $(\varphi_{k+1})_*\eta_{k+1} = \psi_* \circ f_* \circ \chi_*\eta_{k+1} = \psi_*\xi = \lambda_{k+1}$.

The limit map $\varphi\colon C \to A$ obviously satisfies the condition $\varphi_*\eta = \lambda$. The uniqueness property (ii) in Definition 1.11 follows from the uniqueness in Proposition 1.8.

**4.4** THEOREM 4.6. *If* $\dim H^2(L;L) < \infty$, *then the base of the miniversal formal deformation of $L$ is formally embedded in $H^2(L;L)$, that is, it may be described in $H^2(L;L)$ by a finite system of formal equations.*

PROOF. Follows directly from Proposition 4.4.

To make the computation of $C$ more specific, we need an appropriate theory of Massey products.

## 5. Massey products

**5.1.** The obstructions

$$\omega_k\colon H^2_{\mathrm{Harr}}(C_k;\mathbb{K}) \to H^3(L;L)$$

which arise in the construction of the miniversal formal deformation of the Lie algebra $L$ (see 4.1) may be described in terms of Massey products in $H^*(L;L)$. The appropriate theory of Massey products was developed by the second author and Lang [FuL]. We briefly recall this theory.

DEFINITION 5.1. A differential graded Lie algebra (DGLA) is a vector space $\mathcal{C}$ over $\mathbb{K}$ with $\mathbb{Z}$ or $\mathbb{Z}_2$ grading $\mathcal{C} = \bigoplus_i \mathcal{C}^i$ and with commutator operation $\mu\colon L \otimes L \to L$, $\mu(\alpha \otimes \beta) = [\alpha, \beta]$ of degree 0 and a differential $\beta\colon \mathcal{C} \to \mathcal{C}$ of degree $+1$ satisfying the conditions

$$[\alpha, \beta] = -(-1)^{\alpha\beta}[\beta, \alpha],$$
$$\delta[\alpha, \beta] = [\delta\alpha, \beta] + (-1)^{\alpha}[\alpha, \delta\beta],$$
$$[[\alpha, \beta], \gamma] + (-1)^{\alpha(\beta+\gamma)}[[\beta, \gamma], \alpha] + (-1)^{\gamma(\alpha+\beta)}[[\gamma, \alpha], \beta] = 0,$$

where the degree of a homogeneous element is denoted by the same letter as this element.

Our main example of DGLA was introduced in 1.2: $\mathcal{C}^i = C^{i+1}(L;L)$.

The cohomology of $\mathcal{C}$ with respect to $\delta$ is denoted as $\mathcal{H} = \bigoplus_i \mathcal{H}^i$. It is a graded Lie algebra.

**5.2.** The construction of Massey products in $\mathcal{H}$ given below requires the following data. First, a graded cocommutative coassociative coalgebra, that is a $\mathbb{Z}$ or $\mathbb{Z}_2$ graded vector space $F$ over $\mathbb{K}$ with a degree 0 mapping $\Delta\colon F \to F \otimes F$ (comultiplication) satisfying the conditions $S \circ \Delta = \Delta$, where $S\colon F \otimes F \to F \otimes F$ is defined as $S(\varphi \otimes \psi) = (-1)^{\varphi\psi}(\psi \otimes \varphi)$, and $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$. Second, a filtration $F_0 \subset F_1 \subset F$ such that $F_0 \subset \mathrm{Ker}\,\Delta$ and $\mathrm{Im}\,\Delta \subset F_1 \otimes F_1$.

PROPOSITION 5.2 (see [FuL], Proposition 3.1). *Suppose a linear mapping $\alpha\colon F_1 \to \mathcal{C}$ of degree 1 satisfies the condition*

$$\delta\alpha = \mu \circ (\alpha \otimes \alpha) \circ \Delta. \tag{4}$$

16

*Then*

$$\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \operatorname{Ker} \delta.$$

(The right-hand side of the last formula is well defined because $\Delta(F)$ is contained in $F_1 \otimes F_1$, the domain of $\alpha \otimes \alpha$).

DEFINITION 5.3. Let $a: F_0 \to \mathcal{H}$ and $b: F/F_1 \to \mathcal{H}$ be linear maps of degrees 1 and 2. We say that $b$ is contained in the Massey $F$-product of $a$, and write $b \in \langle a \rangle_F$, or $b \in \langle a \rangle$, if there exists a degree 1 linear mapping $\alpha: F_1 \to \mathcal{C}$ satisfying condition (4), and such that the diagrams

$$
\begin{array}{ccc}
F_0 & \xrightarrow{\alpha|_{F_0}} & \operatorname{Ker}\delta \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\
F_0 & \xrightarrow{\ a\ } & \mathcal{H}
\end{array}
\ , \qquad
\begin{array}{ccc}
F & \xrightarrow{\mu\circ(\alpha\otimes\alpha)\circ\Delta} & \operatorname{Ker}\delta \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
F/F_1 & \xrightarrow{\ b\ } & \mathcal{H}
\end{array}
$$

are commutative, where the vertical maps labeled by $\pi$ denote the projections of each space onto the quotient space.

Note that the upper horizontal maps of the diagrams are well defined, since $\alpha(F_0) \subset \alpha(\operatorname{Ker}\Delta) \subset \operatorname{Ker}\delta$ by virtue of (4), and $\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \operatorname{Ker}\delta$ by Proposition 5.2.

Note also that the definition makes sense even in the case, when $F_1 = F$. In this case we do not need to specify any $b$, and we will simply say that $a$ *satisfies the condition of triviality of Massey $F$-products.*

EXAMPLE 5.4. Let $F$ be the dual of the maximal ideal of $\mathbb{K}[t]/(t^{n+1})$, $F_0$ and $F_1$ be the duals of maximal ideals of $\mathbb{K}[t]/(t^2)$ and $\mathbb{K}[t]/(t^n)$. Then $F_0$ and $F/F_1$ are 1-dimensional and are generated respectively by $t$ and $t^n$. In this case $a: F_0 \to \mathcal{H}$ and $b: F/F_1 \to \mathcal{H}$ are characterized by $a(t) \in \mathcal{H}$ and $b(t^n) \in \mathcal{H}$, and it is easy to check that $b \in \langle a \rangle_F$ if and only if $b(t^n)$ belongs to the $n$-th Massey power of $a(t)$ in the classical sense. In particular, for $n = 2$, $b \in \langle a \rangle_F$ if and only if $b(t^2) = [a(t), a(t)]$.

**5.3.** The relationship between Massey products and Lie algebra deformations was established in the article [FuL] by the following result.

Let $A$ be a finite-dimensional local algebra with the maximal ideal $\mathfrak{m}$. Put $F = F_1 = \mathfrak{m}'$ and $F_0 = TA = (\mathfrak{m}/\mathfrak{m}^2)'$.

PROPOSITION 5.5 ([FuL], Theorem 4.2). *A linear map $a: F_0 \to H^2(L; L)$ is a differential of some deformation with base $A$ if and only if $-\frac{1}{2}a$ satisfies the condition of triviality of Massey $F$-products.*

A similar result holds for formal deformations.

## 6. Calculating obstructions

**6.1.** Adopt the notations of 4.1. Consider the sequence

$$\mathbb{K} \xrightarrow{p_1} C_1 \xrightarrow{p_2} \ldots \xrightarrow{p_k} C_k \xrightarrow{\bar{p}_{k+1}} \bar{C}_{k+1}.$$

Recall that all $C_i, \bar{C}_i$ are finite-dimensional algebras,

$$C_1 = \mathbb{K} \oplus H^2(L; L)',$$

and there is an extension

$$0 \to H^2_{\mathrm{Harr}}(C_k; \mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}'_{k+1}} C_k \to 0$$

and an obstruction homomorphism

$$\omega_k \colon H^2_{\mathrm{Harr}}(C_k; \mathbb{K}) \to H^3(L; L).$$

Recall also that

$$C_{k+1} = \bar{C}_{k+1}/\mathrm{Im}\,(\bar{i}_{k+1} \circ \omega'_k).$$

Let $\mathfrak{m}_i, \bar{\mathfrak{m}}_i$ be the maximal ideals in $C_i, \bar{C}_i$. Then we also have the sequence

$$\mathfrak{m}_1 \xrightarrow{p_2} \mathfrak{m}_2 \xrightarrow{p_3} \ldots \xrightarrow{p_k} \mathfrak{m}_k \xrightarrow{\bar{p}_{k+1}} \bar{\mathfrak{m}}_{k+1}.$$

Consider the dual sequence

$$\mathfrak{m}'_1 \xleftarrow{p'_2} \mathfrak{m}'_2 \xleftarrow{p'_3} \ldots \xleftarrow{p'_k} \mathfrak{m}'_k \xleftarrow{\bar{p}'_{k+1}} \bar{\mathfrak{m}}'_{k+1}.$$

This is a sequence of successively embedded cocommutative coassociative coalgebras. Put $\bar{\mathfrak{m}}'_{k+1} = F$, $\mathfrak{m}'_1 = F_0$, $\mathfrak{m}'_k = F_1$. Then

$$F_0 = H^2(L; L), \ F/F_1 = H^2_{\mathrm{Harr}}(C_k; \mathbb{K}).$$

We choose the grading in $F$ to be trivial ($\deg \varphi = 0$ for any $\varphi \in F$).

**6.2.** THEOREM 6.1. $2\omega_k \in \langle \mathrm{id} \rangle_F$ (*this inclusion refers to the Massey product in the sense of Definition 5.3 in the cohomology $\mathcal{H} = \bigoplus_i \mathcal{H}^i$, $\mathcal{H}^i = H^{i+1}(L; L)$, of the DGLA $\mathcal{C} = \bigoplus_i \mathcal{C}^i$, $\mathcal{C}^i = C^{i+1}(L; L)$). Moreover, an arbitrary element of $\langle \mathrm{id} \rangle_F$ is equal to $2\omega_k$ for an appropriate extension of the deformation $\eta_1 = \eta_L$ of $L$ with base $C_1$ to a deformation $\eta_k$ of $L$ with base $C_k$.*

PROOF. The Lie $C_k$-algebra structure $\eta_k$ on $C_k \otimes L$ is determined by the commutators $[l_1, l_2]_{\eta_k} \in C_k \otimes L$ of elements of $L = 1 \otimes L \subset C_k \otimes L$. The difference $[\,,\,]_{\eta_k} - [\,,\,]$ is a linear map $\beta \colon \Lambda^2 L \to \mathfrak{m}_k \otimes L$. This map may be regarded as a map $\mathfrak{m}'_k = F_1 \to \mathrm{Hom}\,(\Lambda^2 L, L) = C^2(L; L)$; we take the last map for $\alpha$ (see Definition 5.3). Obviously, $\alpha|F_0$ represents $a = \mathrm{id} \colon F_0 \to H^2(L; L)$, and the Jacobi identity for $[\,,\,]_{\eta_k}$ means precisely that $\alpha$ satisfies condition (4). Moreover, it is clear, that different $\alpha$'s with these properties correspond precisely to different extensions $\eta_k$ of $\eta_1$.

By definition, a map $b \colon F/F_1 \to H^3(L; L)$ from $\langle a \rangle_F$ is represented by $\mu \circ (\alpha \otimes \alpha) \circ \Delta \colon F \to C^3(L; L)$. On the other hand, the obstruction map $\omega_k \colon H^2_{\mathrm{Harr}}(C_k; \mathbb{K}) = \bar{\mathfrak{m}}'_{k+1}/\mathfrak{m}'_k = F/F_1 \to H^3(L; L)$ is defined by means of lifting the commutator $[\,,\,]_{\eta_k}$ to a skew-symmetric $\bar{C}_{k+1}$-bilinear operation $\{\,,\,\}$ (satisfying some additional conditions – see 3.1). Choose a

basis $m_1, \ldots, m_s$ in $\mathfrak{m}_k$, and extend it to a basis $\bar{m}_1, \ldots, \bar{m}_s, \bar{m}_{s+1}, \ldots, \bar{m}_{s+t}$ of $\bar{\mathfrak{m}}_{k+1}$. (We will also consider the dual bases $\{m'_i\}$ and $\{\bar{m}'_i\}$ in $\mathfrak{m}'_k$ and $\bar{\mathfrak{m}}'_{k+1}$.) Then

$$[l_1, l_2]_{\eta_k} = [l_1, l_2] + \sum_{i=1}^{s} m_i \otimes [l_1, l_2]_i,$$

and the map $\alpha$ acts by the formula

$$\alpha(m'_i)(l_1, l_2) = [l_1, l_2]_i, \quad i = 1, \ldots, , s.$$

We define $\{\,,\,\}$ by the formula

$$\{l_1, l_2\} = [l_1, l_2] + \sum_{i=1}^{s} \bar{m}_i \otimes [l_1, l_2]_i.$$

Let the multiplication in $\bar{\mathfrak{m}}_{k+1}$ be

$$\bar{m}_i \bar{m}_j = \sum_{p=1}^{s+t} c_{ij}^p \bar{m}_p;$$

then $\Delta \colon \bar{\mathfrak{m}}'_{k+1} \to \mathfrak{m}_k \otimes \mathfrak{m}_k$ acts by the formula

$$\Delta(\bar{m}'_p) = \sum_{i,j=1}^{s} c_{ij}^p m'_i \otimes m'_j.$$

We have

$$\{\{l_1, l_2\}, l_3\} = [[l_1, l_2], l_3] + \ldots + \sum_{i,j=1}^{s} \sum_{p=s+1}^{s+t} c_{ij}^p \bar{m}_p \otimes [[l_1, l_2]_i, l_3]_j,$$

where "..." denotes the part corresponding to $\bar{m}_1, \ldots, \bar{m}_s$. Thus the functional $\bar{m}'_p \in \bar{\mathfrak{m}}'_{k+1}$ takes

$$\{\{l_1, l_2\}, l_3\} + \{\{l_2, l_3\}, l_1\} + \{\{l_3, l_1\}, l_2\}$$

into

$$\sum_{i,j=1}^{s} c_{ij}^p \left[ \alpha(m'_j)(\alpha(m'_i)(l_1, l_2), l_3) + \alpha(m'_j)(\alpha(m'_i)(l_2, l_3), l_1) + \alpha(m'_j)(\alpha(m'_i)(l_3, l_1), l_2) \right]$$

$$= \frac{1}{2} \mu \circ (\alpha \otimes \alpha) \circ \Delta(\bar{m}'_p),$$

which shows that $\omega_k = \frac{1}{2} b$. Theorem 6.1 follows.

19

# 7. Further computations

**7.1.** The goal of this Section is to provide a scheme of computation of the base of a miniversal deformation of a Lie algebra, convenient for practical use. We begin with the detailed description of the first two steps of this inductive computation.

As in Section 4, we denote $H^2(L; L)$ by $\boldsymbol{H}$, and also denote by $\mathfrak{m}$ the maximal ideal of the polynomial algebra $\mathbb{K}[\boldsymbol{H}']$. As before, we assume that $\dim \boldsymbol{H} < \infty$. Also we adopt the notations $C_k, \bar{C}_k, \mathfrak{m}_k, \bar{\mathfrak{m}}_k$ of 4.1 and 6.1, and to avoid confusion, we denote the map $\alpha \colon \mathfrak{m}'_k \to C^2(L; L)$ of 6.1 by $\alpha_k$.

According to 4.1,

$$C_1 = \mathbb{K} \oplus \boldsymbol{H}' = \mathbb{K}[\boldsymbol{H}']/\mathfrak{m}^2,$$

and hence

$$\mathfrak{m}_1 = \mathfrak{m}/\mathfrak{m}^2 = \boldsymbol{H}', \ \mathfrak{m}'_1 = \boldsymbol{H}.$$

According to 4.2,

$$\bar{C}_2 = \mathbb{K}[\boldsymbol{H}']/\mathfrak{m}^3,$$

and hence

$$\bar{\mathfrak{m}}_2 = \mathfrak{m}/\mathfrak{m}^3, \ \bar{\mathfrak{m}}'_2 = \boldsymbol{H} \oplus S^2 \boldsymbol{H}.$$

The map

$$\alpha_1 \colon \mathfrak{m}'_1 = \boldsymbol{H} \to C^2(L; L)$$

takes a cohomology class into a representing cocycle. Hence the map

$$\mu \circ (\alpha_1 \otimes \alpha_1) \circ \Delta \colon \bar{\mathfrak{m}}'_2 \to C^3(L; L), \tag{5}$$

where $\Delta \colon \bar{\mathfrak{m}}'_2 \to \mathfrak{m}'_1 \otimes \mathfrak{m}'_1$ is the comultiplication, acts as zero on $\boldsymbol{H}$ (because $\Delta|\boldsymbol{H} = 0$) and takes $\xi\eta \in S^2\boldsymbol{H}$ (where $\xi, \eta \in \boldsymbol{H}$) into the product of the chosen cocycles $\alpha_1(\xi), \alpha_1(\eta)$ representing $\xi, \eta$. Obviously (and according to Proposition 5.2), the image of the map (5) belongs to $\mathrm{Ker}\,\delta$, and the composition of this map with the projection $\pi \colon \mathrm{Ker}\,\delta \to H^3(L; L)$ acts as zero on $\boldsymbol{H}$ and coincides with the multiplication $[\,,\,] \colon S^2\boldsymbol{H} \to H^3(L; L)$ on $S^2\boldsymbol{H}$. Hence

$$\mathfrak{m}'_2 = \boldsymbol{H} \oplus \mathrm{Ker}\,([\,,\,] \colon S^2\boldsymbol{H} \to H^3(L; L)),$$

$$\mathfrak{m}_2 = \frac{\mathfrak{m}}{\mathfrak{m}^3 + J_2}, \ \text{where } J_2 = \mathrm{Im}\,([\,,\,]')$$

$$C_2 = \frac{\mathbb{K}[\boldsymbol{H}']}{\mathfrak{m}^3 + J_2}.$$

Note that if $\dim H^3(L; L) = q$, then $J_2$ is an ideal in $\mathbb{K}[\boldsymbol{H}']$ generated by (at most) $q$ quadratic polynomials.

Furthermore, according to 4.2,

$$\bar{C}_3 = \frac{\mathbb{K}[\boldsymbol{H}']}{\mathfrak{m}^4 + (\mathfrak{m} \cdot J_2)},$$

and hence

$$\bar{\mathfrak{m}}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (\mathfrak{m} \cdot J_2)}, \ \bar{\mathfrak{m}}'_3 = \boldsymbol{H} \oplus S^2\boldsymbol{H} \oplus K,$$

where $K \subset S^3 \boldsymbol{H}$ is the intersection of kernels of the maps

$$f_\varphi \colon S^3 \boldsymbol{H} \to H^3(L; L), \ \varphi \in \boldsymbol{H}',$$

$$f_\varphi(\xi \eta \zeta) = \varphi(\xi)[\eta, \zeta] + \varphi(\eta)[\zeta, \xi] + \varphi(\zeta)[\xi, \eta].$$

The map

$$\alpha_2 \colon \mathfrak{m}'_2 = \boldsymbol{H} \oplus \operatorname{Ker} [\,,\,] \to C^2(L; L)$$

coincides with $\alpha_1$ on $\boldsymbol{H}$ and takes $\sum \xi_i \eta_i \in \operatorname{Ker} [\,,\,] \ (\xi_i, \eta_i \in \boldsymbol{H})$ into a two-dimensional cochain whose coboundary is $\sum [\alpha_1(\xi_i), \alpha_1(\eta_i)]$. Hence the composition

$$\mu \circ (\alpha_2 \otimes \alpha_2) \circ \Delta \colon \bar{\mathfrak{m}}'_3 \to C^3(L; L), \tag{6}$$

where $\Delta \colon \bar{\mathfrak{m}}'_3 \to \mathfrak{m}'_2 \otimes \mathfrak{m}'_2$ is the comultiplication, coincides with the map (5) on $\boldsymbol{H} \oplus S^2 \boldsymbol{H}$ and takes $\sum \xi_i \eta_i \zeta_i$ into

$$[\alpha_1(\xi_i), \alpha_2(\eta_i, \zeta_i)] + [\alpha_1(\eta_i), \alpha_2(\zeta_i, \xi_i)] + [\alpha_1(\zeta_i), \alpha_2(\xi_i, \eta_i)].$$

According to Proposition 2.8, the latter is a cocycle, and the composition of the map (6) and the projection $\pi \colon \operatorname{Ker} \delta \to H^3(L; L)$ acts as zero on $\boldsymbol{H}$, as $[\,,\,]$ on $S^2 \boldsymbol{H}$, and as the "triple Massey product" on $K$. The kernel of this composition is $\mathfrak{m}'_3$, and $\mathfrak{m}_3$ is the dual of this kernel. Thus, by construction,

$$\mathfrak{m}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + J_3}, \ C_3 = \frac{\mathbb{K}[\boldsymbol{H}']}{\mathfrak{m}^4 + J_3},$$

where $J_3 \cap S^2 \boldsymbol{H}' = J_2 \cap S^2 \boldsymbol{H}'$.

**7.2.** Describe now the $k$-th induction step. Suppose that we have already constructed

$$C_k = \frac{\mathbb{K}[\boldsymbol{H}']}{\mathfrak{m}^{k+1} + J_k}, \ \mathfrak{m}_k = \frac{\mathfrak{m}}{\mathfrak{m}^{k+1} + J_k}, \ \alpha_k \colon \mathfrak{m}'_k \to C^2(L; L).$$

Then, according to 4.2,

$$\bar{C}_{k+1} = \frac{\mathbb{K}[\boldsymbol{H}']}{\mathfrak{m}^{k+2} + (\mathfrak{m} \cdot J_k)}, \ \bar{\mathfrak{m}}_{k+1} = \frac{\mathfrak{m}}{\mathfrak{m}^{k+2} + (\mathfrak{m} \cdot J_k)},$$

$$\bar{\mathfrak{m}}'_{k+1} \subset \boldsymbol{H} \oplus S^2 \boldsymbol{H} \oplus \ldots \oplus S^{k+1} \boldsymbol{H}.$$

The image of the composition

$$\mu \circ (\alpha_k \otimes \alpha_k) \circ \Delta \colon \bar{\mathfrak{m}}'_{k+1} \to C^3(L; L), \tag{7}$$

where $\Delta \colon \bar{\mathfrak{m}}'_{k+1} \to \mathfrak{m}'_k \otimes \mathfrak{m}'_k$ is the comultiplication, is contained in $\operatorname{Ker} \delta$ (Proposition 5.2), and the composition

$$\pi \circ \mu \circ (\alpha_k \otimes \alpha_k) \circ \Delta \colon \bar{\mathfrak{m}}'_{k+1} \to H^3(L; L)$$

acts as zero on $\mathfrak{m}'_k$. We put

$$\mathfrak{m}'_{k+1} = \mathrm{Ker}\,(\pi \circ \mu \circ (\alpha_k \otimes \alpha_k) \circ \Delta) \supset \mathfrak{m}'_k.$$

The map $\alpha_k \colon \mathfrak{m}'_k \to C^2(L; L)$ is extended to the map $\alpha_{k+1} \colon \mathfrak{m}'_{k+1} \to C^2(L; L)$ such that $\delta \circ \alpha_{k+1}$ is the restriction of the map (7). The dual to $\mathfrak{m}'_{k+1}$ is

$$\mathfrak{m}_{k+1} = \frac{\mathfrak{m}}{\mathfrak{m}^{k+2} + J_{k+1}},$$

and we put

$$C_{k+1} = \mathbb{K} \oplus \mathfrak{m}_{k+1} = \frac{\mathbb{K}[\boldsymbol{H}']}{\mathfrak{m}^{k+2} + J_{k+1}}.$$

This completes the construction.

**7.3.** Two following useful observations are easily derived from the description of the construction given in 7.1 – 7.2.

PROPOSITION 7.1. *For $l \leq k$,*

$$J_{k+1} \cap S^l \boldsymbol{H}' = J_k \cap S^l \boldsymbol{H}'.$$

PROOF. We use induction with respect to $k$. For $k = 2$ this was proved in 7.1. Suppose that $J_k \cap S^{l-1} \boldsymbol{H}' = J_{k-1} \cap S^{l-1} \boldsymbol{H}'$. Then $(\mathfrak{m} \cdot J_k) \cap S^l \boldsymbol{H}' = (\mathfrak{m} \cdot J_{k+1}) \cap S^l \boldsymbol{H}'$. Hence $\bar{\mathfrak{m}}'_{k+1}$ and $\bar{\mathfrak{m}}'_k$ have the same $S^l \boldsymbol{H}'$ component. Since $\Delta$ has degree 0 with respect to $\boldsymbol{H}'$, and $\alpha_k$ coincides with $\alpha_{k-1}$ on $\mathfrak{m}'_{k-1}$, we may conclude that $\mathfrak{m}'_{k+1}$ and $\mathfrak{m}'_k$ also have the same $S^l \boldsymbol{H}'$ component. Proposition 7.1 follows.

PROPOSITION 7.2. *If $\dim H^3(L; L) = q$, then the ideal $I = \varprojlim I_k = \varprojlim J_k$ from Proposition 4.4 has at most $q$ generators. Less formally, the base of the miniversal deformation of $L$ is the zero locus of a formal map $H^2(L; L) \to H^3(L; L)$.*

PROOF. By construction, $\mathfrak{m}_k = \bar{\mathfrak{m}}_k / G_k$, where $G_k$ is generated by the image of a certain map $\beta_k \colon H^3(L; L)' \to \bar{\mathfrak{m}}_k$ (namely, $\beta_k = (\pi \circ \mu \circ (\alpha_{k-1} \otimes \alpha_{k-1}) \circ \Delta)'$, see 7.2). Actually, $\bar{\mathfrak{m}}_k$ is a quotient of $\bar{\mathfrak{m}}_{k+1}$, and $\beta_{k+1}$ is a lift of $\beta_k$. Put

$$\bar{\mathfrak{m}}_\infty = \varprojlim_{k \to \infty} \bar{\mathfrak{m}}_k, \ \ \mathfrak{m}_\infty = \varprojlim_{k \to \infty} \mathfrak{m}_k.$$

Then $\mathfrak{m}_\infty = \bar{\mathfrak{m}}_\infty / G_\infty$, where $G_\infty$ is generated by the image of

$$\beta_\infty = \varprojlim_{k \to \infty} \beta_k \colon H^3(L; L)' \to \bar{\mathfrak{m}}_\infty.$$

Furthermore,

$$\mathfrak{m}_\infty = \mathfrak{m}/I, \ \ \bar{\mathfrak{m}}_\infty = \mathfrak{m}/(\mathfrak{m} \cdot I),$$

where $I = \varprojlim I_k$. Hence

$$G_\infty = I/(\mathfrak{m} \cdot I),$$

22

and it is clear that generators of $G_\infty$ are lifted to generators of $I$. Since $G_\infty$ is generated by (at most) $q$ generators, Proposition 7.2 follows.

**7.4.** We conclude this section with a brief discussion of the graded case. Suppose that the Lie algebra $L$ is $G$-graded, where $G$ is an Abelian group: $L = \bigoplus_{g \in G} L_g$, $[L_g, L_h] \subset L_{g+h}$. In this case the cochains $\mathcal{C}$ and the cohomology $\mathcal{H}$ get an additional grading: $C^q(L; L) = \bigoplus_{g \in G} C^q_{(g)}(L; L)$ ($\varphi \in C^q_{(g)}(L; L)$, if $\varphi(l_1, \ldots, l_q) \in L_{g_1 + \ldots + g_q - g}$ for $l_1 \in L_{g_1}, \ldots, l_q \in L_{g_q}$), and $H^q(L; L) = \bigoplus_{g \in G} H^q_{(g)}(L; L)$. The condition $\dim H^2(L; L) < \infty$ may be replaced in this case by a weaker condition: $\dim H^2_{(g)}(L; L) < \infty$ for each $g$. We preserve the notation $\boldsymbol{H}$ for $H^2(L; L)$, but $\boldsymbol{H}'$ will denote $\bigoplus_{g \in G} H^2_{(g)}(L; L)'$. All the spaces $\boldsymbol{H}, \boldsymbol{H}', C_k, \mathfrak{m}_k, \bar{\mathfrak{m}}_k, \mathfrak{m}'_k, \bar{\mathfrak{m}}'_k$ have natural $G$-gradings, and all the maps $\alpha_k$ have degree 0. The whole construction is modified correspondingly. We restrict ourselves to the modified version of Proposition 7.2.

PROPOSITION 7.3. *The ideal $I$ from Proposition 4.4 is always generated by homogeneous elements. Moreover, if $\dim H^3_{(g)}(L; L) = q_g$, then $I$ has at most $q_g$ generators of degree $g$. Less formally, the base of the miniversal deformation of $L$ is the zero locus of a formal map $H^2(L; L) \to H^3(L; L)$ of degree 0.*

## 8. Example: Deformations of the Lie algebra $L_1$

**8.1.** Let $L_1$ be the complex Lie algebra of polynomial vector fields $p(x)\dfrac{d}{dx}$ on the line such that $p(0) = p'(0) = 0$. The deformations of this Lie algebra were studied by the first author ([Fi2], [Fi3]), and its formal miniversal deformation was completely described in our joint paper [FiFu]. It turned out that geometrically the base of this deformation is the union of three algebraic curves with a common point: two non-singular, having a common tangent, and one with a cusp, where the tangent at the cusp coincides with the tangent to the smooth components.

Below we show how these results can be obtained by the methods of this article. We will need some (surprisingly little) information about the cohomology and deformations of the Lie algebra $L_1$. All this information is contained in the articles [FeFu], [Fi2], [Fi3], [FiFu].

**8.2.** As a complex vector space, the Lie algebra $L_1$ has the basis $\{e_i | i \geq 1\}$, $e_i = x^{i+1}\dfrac{d}{dx}$, and the commutator operation is $[e_i, e_j] = (j - i)e_{i+j}$. This Lie algebra is $\mathbb{Z}$-graded, $\deg e_i = i$.

PROPOSITION 8.1 ([FeFu], [Fi2]). *The dimensions of $H^2(L_1; L_1)$ and $H^3(L_1; L_1)$ are equal to 3 and 5. Moreover,*

$$\dim H^2_{(q)}(L_1; L_1) = \begin{cases} 1 & \text{if } 2 \leq q \leq 4, \\ 0 & \text{otherwise;} \end{cases}$$

$$\dim H^3_{(q)}(L_1; L_1) = \begin{cases} 1 & \text{if } 7 \leq q \leq 11, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 8.2 ([Fi3]). *Let* $0 \neq \alpha \in H^2_{(2)}(L_1; L_1), 0 \neq \beta \in H^2_{(3)}(L_1; L_1), 0 \neq \gamma \in H^2_{(4)}(L_1; L_1)$. *Then* $0 \neq [\beta, \gamma] \in H^3_{(7)}(L_1; L_1), 0 \neq [\gamma, \gamma] \in H^3_{(8)}(L_1; L_1)$. *Furthermore,* $0 \neq \langle \beta, \beta, \beta \rangle \in H^3_{(9)}(L_1; L_1)$.

The latter means that if $b \in C^2_{(3)}(L_1; L_1)$ is a representative of $\beta$, and if $[b, b] = \delta g, g \in C^2_{(6)}(L_1; L_1)$, then the cohomology class of the cocycle $[b, g] \in C^3_{(9)}(L_1; L_1)$ (which does not depend on the choice of $b$ and $g$) is not equal to 0.

**8.3.** Here are some explicit constructions of deformations of the Lie algebra $L_1$.

PROPOSITION 8.3 ([Fi2]). *The formulas*

$$[e_i, e_j]^1_t = (j - i)(e_{i+j} + te_{i+j-1});$$

$$[e_i, e_j]^2_t = \begin{cases} (j - i)e_{i+j} & \text{if } i \neq 1, j \neq 1, \\ (j - 1)e_{j+1} + tje_j & \text{if } i = 1, j \neq 1; \end{cases}$$

$$[e_i, e_j]^3_t = \begin{cases} (j - i)e_{i+j} & \text{if } i \neq 2, j \neq 2, \\ (j - 2)e_{j+2} + tje_j & \text{if } i = 2, j \neq 2 \end{cases}$$

*determine three one-parameter deformations of the Lie algebra $L_1$. All the three deformations are pairwise not equivalent. Moreover, if $L^1_1, L^2_1, L^3_1$ are Lie algebras from the three families corresponding to arbitrary non-zero values of the parameter (up to an isomorphism, they do not depend on the non-zero parameter value), then neither two of $L^1_1, L^2_1, L^3_1$ are isomorphic to each other.*

COROLLARY 8.4. *The base of any versal deformation of the Lie algebra $L_1$ contains at least three different irreducible curves.*

**8.4.** We will use the notations of Section 7. Let $\alpha, \beta, \gamma$ be a basis of $\boldsymbol{H} = H^2(L_1; L_1)$ (as in Proposition 8.2), and let $x, y, z$ be the dual basis in $\boldsymbol{H}'$. The algebra $S^* \boldsymbol{H}' = \mathbb{C}[x, y, z]$ has the monomial basis $\{x^p y^q z^r\}$. Let $\{\alpha^p \beta^q \gamma^r\}$ be the dual basis in the coalgebra $S^* \boldsymbol{H}$; the comultiplication $\Delta \colon S^* \boldsymbol{H} \to S^* \boldsymbol{H} \otimes S^* \boldsymbol{H}$ acts by the formula

$$\Delta(\alpha^p \beta^q \gamma^r) = \sum_{i=0}^p \sum_{j=0}^q \sum_{k=0}^r \alpha^i \beta^j \gamma^k \otimes \alpha^{p-i} \beta^{q-j} \gamma^{r-k}.$$

Choose cocycles $a \in C^2_{(2)}(L_1; L_1), \ b \in C^2_{(3)}(L_1; L_1), \ c \in C^2_{(4)}(L_1; L_1)$ representing $\alpha, \beta, \gamma$. Then

$$\alpha_1 \colon \mathfrak{m}'_1 = \boldsymbol{H} \to C^2(L_1; L_1)$$

is defined by the formulas

$$\alpha_1(\alpha) = a, \ \alpha_1(\beta) = b, \ \alpha_1(\gamma) = c.$$

Since $H^3_{(q)}(L_1; L_1) = 0$ for $q < 7$, there exist $d \in C^2_{(4)}(L_1; L_1), e \in C^2_{(5)}(L_1; L_1), f \in C^2_{(6)}(L_1; L_1), g \in C^2_{(6)}(L_1; L_1)$, such that $[a, a] = \delta d, [a, b] = \delta e, [a, c] = \delta f, [b, b] = \delta g$ (the notation $g$ has been already used in 8.2). Since $c \in C^2_{(4)}(L_1; L_1)$ is a cocycle, we can replace

$d$ with $d + tc$, where $t$ is an arbitrary complex number. Finally, since $\delta[a, d] = 0$, we also have $[a, d] = \delta h$ for some $h \in C^2_{(6)}(L_1; L_1)$.

The space $\bar{\mathfrak{m}}'_2 = \boldsymbol{H} \oplus S^2 \boldsymbol{H}$ is spanned by $\alpha, \beta, \gamma, \alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \beta\gamma, \gamma^2$. The map $\mu \circ (\alpha_1 \otimes \alpha_1) \circ \Delta \colon \bar{\mathfrak{m}}'_2 \to C^3(L_1; L_1)$ acts in the following way:

$$\alpha, \beta, \gamma \mapsto 0; \ \alpha^2 \mapsto \delta d, \ \alpha\beta \mapsto 2\delta e, \ \alpha\gamma \mapsto 2\delta f, \ \beta^2 \mapsto \delta g,$$

$$\beta\gamma \mapsto 2[b, c] \notin \operatorname{Im} \delta, \gamma^2 \mapsto [c, c] \notin \operatorname{Im} \delta.$$

Hence $\mathfrak{m}'_2$ is generated by $\alpha, \beta, \gamma, \alpha^2, \alpha\beta, \alpha\gamma, \beta^2$, and

$$\alpha_2 \colon \mathfrak{m}'_2 \to C^2(L_1; L_1)$$

is defined as $\alpha_1$ on $\boldsymbol{H}$ and

$$\alpha_2(\alpha^2) = d + tc, \ \alpha_2(\alpha\beta) = 2e, \ \alpha_2(\alpha\gamma) = 2f, \ \alpha_2(\beta^2) = g.$$

Furthermore,

$$\mathfrak{m}_2 = \frac{\mathfrak{m}}{\mathfrak{m}^3 + (yz, z^2)},$$

$$\bar{\mathfrak{m}}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (\mathfrak{m} \cdot (yz, z^2))} = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (xyz, xz^2, y^2z, yz^2, z^3)},$$

and

$$\bar{\mathfrak{m}}'_3 = \boldsymbol{H} \oplus S^2 \boldsymbol{H} \oplus K,$$

where $K$ is the subspace of $S^3 \boldsymbol{H}$ spanned by $\alpha^3, \alpha^2\beta, \alpha^2\gamma, \alpha\beta^2, \beta^3$. The map $\mu \circ (\alpha_2 \otimes \alpha_2) \circ \Delta \colon \bar{\mathfrak{m}}'_3 \to C^3(L_1; L_1)$ acts as $\mu \circ (\alpha_1 \otimes \alpha_1) \circ \Delta$ on $\boldsymbol{H} \oplus S^2 \boldsymbol{H}$ (see above), and acts on $K$ in the following way:

$$\alpha^3 \mapsto 2[a, d + tc] = \delta(h + tf),$$
$$\alpha^2\beta \mapsto 4[a, e] + 2[b, d] + 2t[b, c],$$
$$\alpha^2\gamma \mapsto 4[a, f] + 2[c, d] + 2t[c, c],$$
$$\alpha\beta^2 \mapsto 2[a, g] + 4[b, e],$$
$$\beta^3 \mapsto 2[b, g] \notin \operatorname{Im} \delta.$$

Since $4[a, e] + 2[b, d] \in \operatorname{Ker} \delta$, $[b, c] \notin \operatorname{Im} \delta$, and $\dim H^3_{(7)}(L_1; L_1) = 1$, we can choose $t$ in such a way that the image of $\alpha^2\beta$ is cohomologous to $0$,

$$\alpha^2\beta \mapsto \delta k, \ k \in C^2_{(7)}(L_1; L_1).$$

Since $4[a, f] + 2[c, d] + 2t[c, c], 2[a, g] + 4[b, e] \in \operatorname{Ker} \delta$, $[c, c] \notin \operatorname{Im} \delta$, and $\dim H^3_{(8)}(L_1; L_1) = 1$, there exist complex numbers $A, B$ such that the images of $\alpha^2\gamma - A\gamma^2$, $\alpha\beta^2 - B\gamma^2$ are cohomologous to $0$,

$$\alpha^2\gamma - A\gamma^2 \mapsto \delta l, \ l \in C^2_{(8)}(L_1; L_1),$$
$$\alpha\beta^2 - B\gamma^2 \mapsto \delta m, \ m \in C^2_{(8)}(L_1; L_1).$$

Hence $\mathfrak{m}_3'$ is generated by the generators of $\mathfrak{m}_2'$ (see above) and also $\alpha^3, \alpha^2\beta, \alpha^2\gamma - A\gamma^2$, $\alpha\beta^2 - B\gamma^2$. Thus

$$\mathfrak{m}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (yz, z^2 + Ax^2z + Bxy^2, y^3)}.$$

To complete this description of the base of the miniversal deformation of $L_1$, we need to continue the induction to calculate $\mathfrak{m}_4$ and $\mathfrak{m}_5$. This would require more information about the multiplications in the cohomology of $L_1$. It turns out, however, that we can avoid any additional computations if we use Corollary 8.4.

**8.5.** According to Propositions 7.3 and 8.1, the base of the miniversal deformation of $L_1$ is $\mathbb{C}[[x, y, z]]/(F_1, F_2, F_3, F_4, F_5)$, where $F_1, \ldots, F_5$ are polynomials in $x, y, z$ of degrees $7, \ldots, 11$ (with $\deg x = 2, \deg y = 3, \deg z = 4$). The calculations of 8.4 show that

$$\begin{aligned}
F_1 &= yz + \ldots, \\
F_2 &= z^2 + Ax^2z + Bxy^2 + \ldots, \\
F_3 &= y^3 + \ldots,
\end{aligned}$$

where "$\ldots$", and $F_4, F_5$ as well, are linear combinations of 4- and 5-fold products of $x, y, z$ having appropriate degrees. These products are the following monomials.

$$\begin{aligned}
&\text{degree 7: none,} \\
&\text{degree 8: } x^4, \\
&\text{degree 9: } x^3y, \\
&\text{degree 10: } x^5, x^3z, x^2y^2, \\
&\text{degree 11: } x^4y, x^2yz.
\end{aligned}$$

We exclude the monomial $x^2yz$, because it can be extinguished by adding a constant times $x^2F_1$, and get the following intermediate result.

LEMMA 8.5. *The base of the miniversal deformation of $L_1$ is described in $H^2(L_1; L_1)$ by a system of formal equations*

$$\begin{aligned}
\beta\gamma &= 0, \\
\gamma^2 + A\alpha^2\gamma + B\alpha\beta^2 + C\alpha^4 &= 0, \\
\beta^3 + D\alpha^3\beta &= 0, \\
E\alpha^5 + F\alpha^3\gamma + G\alpha^2\beta^2 &= 0, \\
H\alpha^4\beta &= 0.
\end{aligned} \tag{8}$$

Consider the zero locus $X$ of the first three equations (8).

LEMMA 8.6. *If $C = BD$, $A^2 \neq 4C$, and $D \neq 0$, then $X$ is the union of three irreducible curves. Otherwise $X$ does not contain three different irreducible curves.*

PROOF. Let $(\alpha, \beta, \gamma) \in X$. The first equation (8) says that either $\beta = 0$, or $\gamma = 0$. If $\beta = 0$, then the third equation holds, and the second equation becomes

$$\gamma^2 + A\alpha^2\gamma + C\alpha^4 = (\gamma + u\alpha^2)(\gamma + v\alpha^2) = 0, \tag{9}$$

where $u \neq v$ if $A^2 \neq 4C$. Hence $X \cap \{\beta = 0\}$ is the union of two parabolas. If $\gamma = 0$, then the second and the third equations become

$$\alpha(B\beta^2 + C\alpha^3) = 0,$$
$$\beta(\beta^2 + D\alpha^3) = 0,$$

which describes just one point $\alpha = 0, \beta = 0$ if $C \neq BD$, the semicubic parabola $\beta^2 + D\alpha^3 = 0$ if $0 \neq C = BD$, and the union of the same semicubic parabola and the line $\beta = 0$ if $0 = C = BD$. In the last case one of the curves (9) is also the line $\beta = 0, \gamma = 0$. Lemma 8.6 follows.

THEOREM 8.7. *The base of the miniversal deformation of the Lie algebra $L_1$ is described in $H^2(L_1; L_1)$ by the system of formal equations*

$$\beta\gamma = 0,$$
$$\gamma^2 + A\alpha^2\gamma + B\alpha(\beta^2 + D\alpha^3) = 0,$$
$$\beta(\beta^2 + D\alpha^3) = 0,$$

*where $A^2 \neq 4BD$, and $D \neq 0$.*

PROOF. Corollary 8.4 and Lemma 8.6 imply that in equations (8) $C = BD, A^2 \neq 4C$, and $D \neq 0$. Hence the three curves, which are contained in the base of the miniversal deformation according to Corollary 8.4, are

$$\beta = 0, \ \gamma + u\alpha^2 = 0;$$
$$\beta = 0, \ \gamma + v\alpha^2 = 0;$$
$$\gamma = 0, \ \beta^2 + D\alpha^3 = 0,$$

where $u \neq v$, $u + v = A$, $uv = BD$. Hence the left hand sides of the last two equations (8) should be equal to 0 on these curves. The monomial $\alpha^4\beta$ is not equal to 0 on the third of the curves; hence $H = 0$. If $\beta = 0$, then the fourth equation becomes $\alpha^3(E\alpha^2 + F\gamma) = 0$, which cannot hold on *both* parabolas $\gamma + u\alpha^2 = 0$, $\gamma + v\alpha^2 = 0$ unless $E = F = 0$. Finally, if $\gamma = 0$, then the fourth equation (with $E = F = 0$) becomes $G\alpha^2\beta^2 = 0$ which does not hold on the third curve unless $G = 0$.

**8.6.** Note that the computations made in the article [FiFu] let us find the constants $A, B, D$ from Theorem 8.7. Since these constants depend on a particular choice of cocycles $a \in C^2_{(2)}(L_1; L_1)$, $b \in C^2_{(3)}(L_1; L_1)$, $c \in C^2_{(4)}(L_1; L_1)$ representing generators of $H^2(L_1; L_1)$, we need to specify these cocycles first.

Let $W$ be an $L_1$-module spanned by $e_j$ with all $j \in \mathbb{Z}$ and with the $L_1$-action $e_i(e_j) = (j - i)e_{i+j}$. It is an extension of the adjoint representation. Define a cochain

$$\mu_k \in C^1_{(k)}(L_1; W), \ k \geq 2,$$

by the formula

$$\mu_k(e_i) = (-1)^{i+1} \binom{k-1}{i-2} e_{i-k}.$$

PROPOSITION 8.8 [FiFu]. *If* $k = 2, 3, 4$, *then* $\delta\mu_k$ *belongs to* $C^2_{(k)}(L_1; L_1)$ *and is a cocycle not cohomologous to* 0.

PROPOSITION 8.9 [FiFu]. *If one chooses* $a, b, c$ *to be* $\delta\mu_2$, $\delta\mu_3$, $\delta\mu_4$, *then*

$$A = -\frac{2 \cdot 11 \cdot 37}{5 \cdot 13^2}, \ B = \frac{4 \cdot 7 \cdot 17}{3 \cdot 25 \cdot 13}, \ D = \frac{32 \cdot 27}{13^3}.$$

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